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THE CONSTRUCTION OF CONICS UNDER GIVEN CONDITIONS.

By B. M. WOODS, University of California.

The following discussion treats by synthetic methods the matter presented by Professor M. W. Haskell[1] in a paper of the same title, where the treatment is for the most part analytic.

SECTION I. CONICS DEFINED BY FIVE POINTS OR BY FIVE TANGENTS.

Theorem I. *The eight vertices of two quadrangles with the same diagonal triangle lie on a conic.*

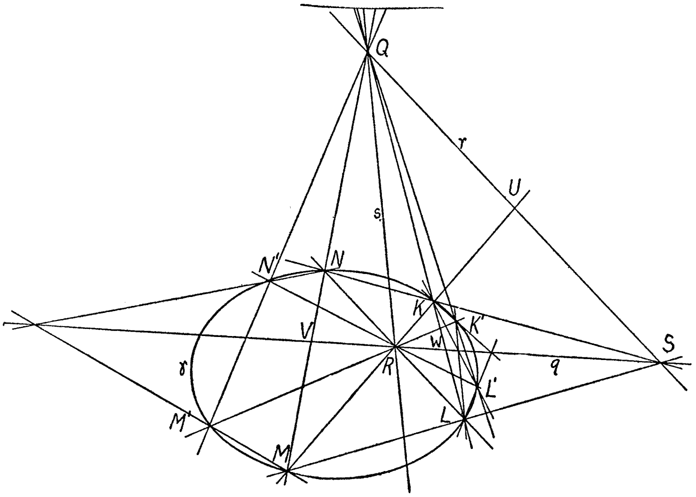

Fig. 1.

Consider a point Q (Fig. 1) and its polar q with respect to a given conic γ. If a point R is selected on q, its polar r passes through Q and cuts q in a point S, whose polar s is the line QR. A self-polar triangle QRS is thus determined.

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[1] *Bulletin of the Amer. Math. Soc.*, 2d Series, Vol. XI, No. 5, pp. 268–273.

If now an arbitrary ray QK be drawn through Q, cutting the conic in K and L, and the rays KR and LR be drawn intersecting the conic in M and N respectively, the ray QN will pass through M. For, by the fundamental properties of the polar, Q, K, W, L are four harmonic points.[1] QN cuts the four harmonic rays QR, KR, WR, and LR in four harmonic points. Also the fourth harmonic of N with respect to Q and V must be on the conic. Hence, QN meets KR on the conic. Similarly NK and ML may be shown to intersect at S. Hence, an arbitrary ray through Q determines uniquely a quadrangle $KLMN$ with its vertices on the conic γ and with the diagonal triangle QRS.

Now a quadrangle is uniquely determined by its diagonal triangle and one vertex as follows: In Fig. 2, let QRS be the diagonal triangle and M the given vertex. Join M to Q, R, and S. Construct U, the harmonic conjugate of V (the intersection point of MS and QR) with respect to Q and R. Join U to S, determining N and K. Draw NR and KQ. This construction gives the quadrangle $KLMN$ desired, for, from the properties of the quadrangle, the two sides passing through S cut QR in points that are harmonic conjugates of Q and R.

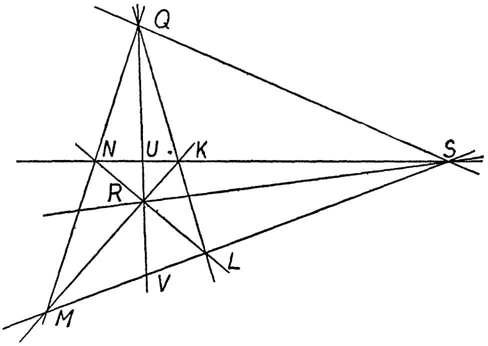

Fig. 2.

If, then, the diagonal triangle QRS is given, one vertex K determines the quadrangle $KLMN$. If another vertex K' be assigned arbitrarily, it determines with K, L, M, N a conic γ. It also determines a quadrangle $K'L'M'N'$ with diagonal triangle QRS and vertices on the conic γ. Since these two quadrangles are chosen arbitrarily among those that have the diagonal triangle QRS, the theorem is established.

From this follows a simple construction by points of a conic of which five points are given.

Construction: Construct the complete quadrangle of any four of the points with its diagonal triangle. The fifth point and the diagonal triangle obtained determine a second quadrangle whose vertices lie on the required conic. By taking the quadrangle of each four points in turn, we locate fifteen new points in this manner. Combining these in various ways, we may locate as many points as desired.

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[1] See Reye, "Geometrie der Lage," Chapter VIII.

We shall omit here and in general in what follows, all theorems obtained by dualization.

Theorem II. *The lines joining pairs of vertices of two quadrangles with the same diagonal triangle form a quadrilateral with this diagonal triangle.*

Let us consider Fig. 1 again. QR was chosen originally as an arbitrary ray through Q. Hence, the demonstration of Theorem I includes the proof that, if QK and QK' be arbitrary rays through Q, KK' and LL' will meet on q, the polar of Q. So, also, will $K'L$ and KL'. Hence, in the quadrilateral formed by KK', LL', MM', and NN', we have KK' and LL' intersecting on q and similarly MM' and NN'. Likewise, we may show that KK' and NN', and LL' and MM' intersect on s, and finally KK' and MM', and LL' and NN' on r. Hence, the diagonal triangle of the quadrilateral is QRS, the same as that of either quadrangle $KLMN$ or $K'L'M'N'$.

If now K' approaches K along the conic, L', M', and N' approach L, M, and N respectively and the sides of the quadrilateral approach as limiting positions, the tangents to the conic at the vertices of the quadrangle. From which follows

Theorem III. *The tangents at any four points of a conic form a quadrilateral which has the same diagonal triangle as the quadrangle of those four points.*

By combining this theorem with the dual of Theorem I, viz., the eight sides of two quadrilaterals with the same diagonal triangle envelop a conic, we obtain the following

Theorem IV. *The eight tangents at the points of intersection of two conics have the same diagonal triangle as the quadrilateral of the four tangents to either one of the conics, at these points. These two quadrilaterals have the same diagonal triangle and, consequently, their sides envelop a conic.*

The second part of the theorem is the dual of the first. The conic mentioned in the second part is the covariant conic[1] of the two given conics, and is the locus of points from which the pairs of tangents to the given conics are harmonic conjugates of each other. A synthetic discussion has been given by Professor Lehmer.[2]

Theorem V. *The quadrilateral of the four common tangents to two conics has the same diagonal triangle as the quadrangle of the intersection points of the two conics.*

For let PP' be one of these common tangents, P and P' being the respective points of contact. Then if Q, R, S; Q', R', S', respectively, be the remaining vertices of the quadrangles determined by P and P', these quadrangles having the same diagonal triangle as the quadrangle of the four points of intersection of the two conics, then QQ', RR', and SS' will be the remaining sides of the quadrilateral determined by PP', having the same diagonal triangle as the quadrangles just determined. Q, R, S will lie on the first conic; Q', R', S' on the second; and QQ', RR', SS' will be tangent to both.

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[1] See Salmon, *Conic Sections*, pp. 306 and 344.

[2] *Amer. Math. Monthly*, Feb., 1908, "A Discussion by Synthetic Methods of the Covariant Conic of two Given Conics."

Section 2. Involutory Quadratic Transformation. Conics Through Four Given Points and Tangent to a Given Line.

An involution of rays may be regarded as composed of two superposed projective pencils of rays where the correspondence is such that a given ray has but one corresponding ray whether it be considered as belonging to one or the other of the superposed pencils. Consider two such involutions of rays with centers at A and B (Fig. 3), to be called the involutions A and B. To a point Q of the plane corresponds a point Q' obtained by finding the point of intersection of AQ' and BQ', the rays at A and B corresponding to AQ and BQ respectively. If a point Q moves along a ray q, it projects to A and B in two perspective pencils of rays. The corresponding rays AQ' and BQ' in the involutions A and B describe pencils of rays projective to the pencils AQ and BQ respectively and hence projective to each other. Q', therefore, describes a conic through A and B. It has been noted that to every point Q of the plane there is a corresponding point Q', and conversely. There are, however, special points and lines for which the correspondence is not unique. Any point of the line AB, for instance, corresponds to the same point C, the point of intersection of the rays at A and B corresponding to AB considered first as a ray of the involution A and then as one of the involution B. Conversely, AB may be considered as corresponding to C.

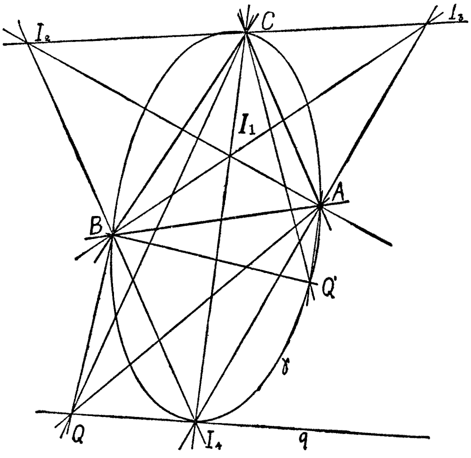

Fig. 3.

Denote now by γ the conic described by Q', as Q describes q. As Q' describes γ, the pencils of rays CQ' and CQ are projective to each other. For the pencil CQ' is projective to the pencil AQ' and the pencil CQ is perspective to the pencil AQ, the pencils AQ and AQ' are projective to each other by hypothesis. Moreover, Q and Q' are related to each other reciprocally by means of the involutions A and B. Hence, the pencils CQ' and CQ, superposed at C, give an involution of rays at that point which may be paired with the involution A or B to give the original transformation. Suppose that A and B, which were chosen arbitrarily, have double rays. The intersection point of a double ray of A with a double ray of B is a self-corresponding point in this transformation of the plane. Hence, it must also lie on a double ray of C. Therefore, the involution C has double rays, and two of the intersections of the double rays of A and B lie on one of them, while the other two lie on the other. Hence, the six double rays of A, B, and C intersect in four points and, consequently, form a quadrangle whose diagonal triangle is ABC. Denote the four intersection points by I_1, I_2, I_3, I_4. These are obviously the only self-corresponding points of the plane. It should also be noted that since the involution C has been added, all the vertices of the triangle ABC correspond to its opposite sides, and conversely.[1]

Now, if with the involutions A and B arbitrarily chosen and the involution at C determined as above, we consider the point Q moving on an arbitrary ray q, its corresponding point describes a conic γ, through A, B, and C. If, as Q describes q, Q' crosses q, Q and Q' are at that instant the points of intersection of γ and q. Hence the

Theorem VI. *The conic into which a line is converted by the involutory quadratic transformation meets this line in a pair of corresponding points, the only pair of such points on the line. If the line is tangent to its corresponding conic, the point of tangency is a self-corresponding point of the transformation.*

Corollary. The conic corresponding to a line through a self-corresponding point is the conic through A, B, C which is tangent to the given line at the self-corresponding point. The conic of Fig. 3 is drawn in this position for convenience.

Moreover, any two corresponding rays of an involution are harmonic conjugates of the double rays. Hence, the rays joining Q and Q' to A, B, and C are harmonic conjugates of the double rays at those points. Since the pencil of conics through the self-corresponding points I_1, I_2, I_3, I_4 cuts out an involution of points on any ray of the plane, in particular on q, the double points of this involution mark the points of tangency of the two conics of the pencil tangent to the line. The three pairs of opposite sides of the quadrangle of the self-corresponding points are the pairs of double rays of the three involutions A, B, and C; and they cut q in three pairs of points of the involution. Now, since Q and Q' on q are harmonic conjugates of each pair of these points, they are the double points sought for. Whence, we write

Theorem VII. *The line joining a pair of corresponding points, Q and Q', in an involutory quadratic tranformation of the plane, is tangent to two conics of the pencil of conics through the four self-corresponding points of the transformation, the points of tangency being Q and Q'.*

Conversely, *if there are two conics of this pencil of conics tangent to an arbitrary line of the plane, the points of tangency are corresponding points in the transformation.*

The reality of the conics of the pencil through I_1, I_2, I_3, I_4, tangent to a given line, depends on whether the involution of points on that line determined by the

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[1] For a more complete discussion of this transformation see Professor Lehmer's article in AMER. MATH. MONTHLY, "On the Combination of Involutions," March, 1911.

pencil of conics has double points or not. The intersections of the line with the opposite sides of the quadrangle give pairs of corresponding points on the line. These pairs of points must not divide each other if the involution is to have double points. Therefore, the given line must cut each pair of opposite sides of the quadrangle of the four points either in the segments upon which a vertex of the diagonal triangle lies, or else in those segments upon which it does not lie. If this condition is fulfilled for one pair of sides, it is fulfilled for the remaining pairs.

In this transformation, the points of intersection of the given line with the sides of the quadrangle of the self-corresponding points are points on the double rays of A, B, and C. A point on one of these double rays is evidently converted into a point on that same ray,—its harmonic conjugate with respect to the two self-corresponding points on the ray.

Theorem VIII. *The conic into which a given line is converted by the involutory quadratic transformation passes through the harmonic conjugates of the intersections of that line with the six sides of the quadrangle whose vertices are* I_1, I_2, I_3, I_4.

The conic corresponding to the line at infinity is then the conic passing through the middle points of the sides of the quadrangle and through the vertices of its diagonal triangle.

It is to be noted that a pair of corresponding points, Q and Q', together with the vertices of the diagonal triangle, completely determine the quadratic transformation, as follows: Q and Q' must project to A (Fig. 3), for example, in a pair of corresponding rays. Likewise, the rays AB and AC are corresponding. This determines the involution at A. Similarly, at B and C. The self-corresponding points may be found as intersections of the double rays of the several involutions.

SECTION 3. CONICS WITH THREE TANGENTS AND TWO POINTS GIVEN.

The proof for the number of conics with three points and two tangents given has been established by Steiner.[1] The following proof is carried out more or less along the same lines. The problem at hand is the number and construction of the conics fulfilling the given conditions, the solution of which depends on an auxiliary theorem, which will be discussed first.

Theorem IX. *The range of conics tangent to four lines determines at any point of the plane an involution of rays composed of the pairs of tangents from the points to the conics of the range. If the four fixed tangents should coincide in pairs so that the range of conics consists of those tangent to two fixed lines with fixed points of contact, the involution at any point of the plane has double rays.*

For, in particular, the line joining this point to the point of intersection of the two fixed lines is a double ray of the involution. This follows from the fact that the three pairs of opposite vertices of the complete quadrilateral of the four fixed lines, being the degenerate conics of the range, project to any point S in three pairs of corresponding rays of the involution determined at S. In our case, two pairs of opposite vertices have fallen together in one point, the point of intersection of the fixed rays, which therefore projects as a double ray.

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[1] *Vorlesungen über Synthetische Geometrie,* Zweiter Teil, p. 236, ed. 1876.

Apply this to the solution of the problem. Call the given lines a, b, and c; the opposite vertices of the triangle they determine A, B, and C; and the given points L and L' (Fig. 4). An involution of rays is determined at A, for instance, by the rays b and c as one pair and LA and $L'A$ as the other. Similarly, at B and C. Call the double rays of the involution at A, g and h; of the involution at B, g' and h'; of the involution at C, g'' and h''. Since this method of establishing the involutions determines an involutory quadratic transformation of the plane which has L and L' for corresponding points and has the centers of its involutions at A, B, and C, the intersection points of the double rays at A, B, and C will be the four vertices of a complete quadrangle with diagonal triangle ABC. Call these intersection points I_1, I_2, I_3, I_4. They are naturally the self-corresponding points of the transformation. I_1 is the intersection point of double rays of A, B, and C; say, g, g', and g''. The conics tangent to I_1L and I_1L' at L and L', respectively, will determine an involution of rays at A, as at any point of the plane, made up of the tangents from A to the respective conics. g is a double ray of this involution, and AL and AL' are corresponding rays. For g joins A to the point of intersection of the fixed tangents to the range of conics, and L and L' constitute one of the degenerate conics of the range. But this is the involution previously determined at A. Likewise, at B and C. Now, in this involution at A, b and c are corresponding rays; at B, a and c are corresponding rays. Hence, the particular conic of this range of those tangent to I_1L at L and I_1L' at L' which is itself tangent to c, is also tangent to a and b. Since it also passes through L and L', it is one of the conics required. Each of the four self-corresponding points gives one such conic; viz., the conic tangent to the rays joining it to L and L' at L and L', and tangent to the three rays a, b, and c. In each case, LL' is the polar of the self-corresponding point joined to L and L'.

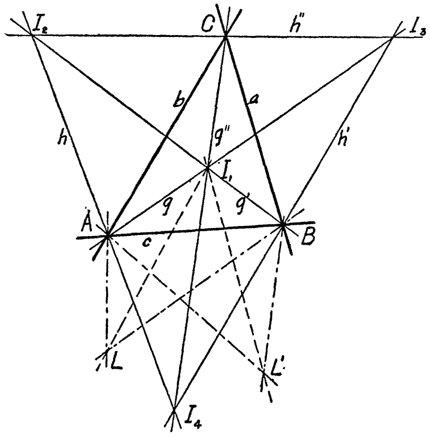

Fig. 4.

Hence, the following theorems, which are grouped for convenience.

Theorem X. *There are four conics tangent to three given rays and passing* *through two given points. The ray joining the two given points being a common chord, the conics are determined as soon as its poles are known. Its four poles are the self-corresponding points of the involutory quadratic transformation of which the given lines constitute the fundamental triangle, and by which the given points are interchanged.*

These conics will all be real if the involutions determined at A, B, and C have double rays. For this to happen, the rays joining A, for instance, to L and L' must not be divided by b and c. Hence, all the conics will be real if L and L' lie together in one of the four sections into which the given lines divide the plane[1] (Fig. 5). Otherwise, they are all imaginary, since two of the involutions have no double rays.

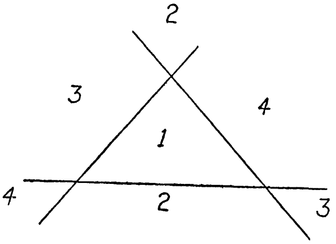

FIG. 5.

A SIMPLE METHOD OF CONSTRUCTING THE NORMALS TO A PARABOLA.

By S. G. BARTON, University of Pennsylvania.

We know by Harvey's Theorem that the feet of the three normals to a parabola from any point lie on a circle which passes through the vertex. The converse is also true, the normals whose feet lie on a circle through the vertex are concurrent. For let (d, e) be the center of such a circle. Its equation then is $x^2 + y^2 - 2dx - 2ey = 0$, which intersects the parabola $y^2 = 4px$ in points whose ordinates are given by

$$y^3 - 8p(d - 2p)y - 32p^2 e = 0.$$

Since this lacks the second term the sum of its roots is zero which is the condition for concurrent normals.

The equation giving the ordinates of the feet of the normals through a point (h, k) is

$$y^3 - 4p(h - 2p)y - 8p^2 k = 0.$$

But these equations must be equivalent, hence by equating coefficients $2d - 4p = h - 2p$ and $32e = 8k$; therefore $d = \frac{1}{2}h + p$ and $e = \frac{1}{4}k$.

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[1] See Annie Dale Biddle, "Constructive Theory of the Unicursal Plane Quartic by Synthetic Methods," Univ. Calif. Publ. Math., Vol. 1, No. 2.